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## Alignment in the fully asymmetric SK model

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**Abstract.** The long-time limit of the alignment function (or remanent energy  $E = -\langle \sigma h \rangle / 2$ ) of the fully asymmetric SK model is investigated analytically for parallel ( $E_p$ ), sequential ( $E_s$ ) and random-sequential ( $E_{rs}$ ) update. As expected we find  $E_p > E_s > E_{rs}$ . The results are supported by numerical simulations.

### 1. Introduction

Several years ago disordered spin models with asymmetric couplings gained growing interest due to their relevance to neural networks. In general these models represent a certain class of systems in non-equilibrium statistical physics which cannot be described by means of a Hamiltonian. This fact causes several difficulties in treating them analytically. So, apart from huge numerical simulations, few exact results are known.

The introduction of asymmetric couplings in the SK model [1] leads to a generalization with directed couplings  $J_{ij} \neq J_{ji}$  [2]. In general the couplings can be distributed completely independently (fully asymmetric case) or correlated by pairs. The problems arising in calculating the properties of these systems are due to the fact that no detailed balance condition holds and therefore no fluctuation-dissipation theorems exist. The stationary distribution has to be calculated directly by solving the dynamical equations. This turns out not to be possible in most cases, except in the fully asymmetric model, where some exact results are known [3, 4].

Numerical simulations of the general asymmetric case with finite correlations between  $J_{ij}$  and  $J_{ji}$  seem to indicate a transition from a state with vanishing to a state with finite remanent magnetization (i.e. the overlap of the initial and the final state) varying the degree of the asymmetry [5, 6]. In this range only an expansion around the fully-asymmetric point is known [7], where the remanent magnetization is zero. This transition point itself remains controversial since older investigations of analogous systems with continuous spins claimed that for any finite degree of asymmetry the spin-glass state will be destroyed [2, 8].

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In this paper we want to calculate the alignment ('remanent energy') in the fully-asymmetric model for three different kinds of dynamics (parallel, random sequential and sequential). The remanent magnetization vanishes in this model for all three dynamics, a result which can be shown to hold exactly at least for random sequential update. The alignment function can indeed distinguish between them and is thus a quantity which is able to measure the influence of the update rule on the dynamics. The limiting state of the system is not known in detail, but the expectation value of the alignment function approaches a stationary value as can be seen below.

There have been some attempts to analyse the influence of the update rule on the attractors in asymmetric spin glasses [9] and automata networks [10]. They show that the stability of the metastable states is not affected by a change of the update rule, but the basins of attraction can vary drastically. For deterministic dynamics (and also for special cases of random update [10]) cycles occur, in which always some spins do not point in the direction of their local fields. The alignment function is by definition a measure of this spin-field correlation and the occurrence of cycles will thus increase it. It is known that, in the asymmetric SK model, parallel update yields an exponent of the number of cycles of length two which is twice as large as the exponent for the number of metastable states, both increasing exponentially with the total number of spins [9]. Its alignment function is the highest one as will be shown below.

We will use three different methods to calculate the long-time limits for the different dynamical rules. The simplest case, parallel update, can be treated by simple symmetry arguments. This result turns out to be independent of the system size and the topology of the underlying lattice. For random sequential update the well known path-integral formulation by Sommers is used, a powerful method in the derivation of almost all known exact results for this model. In this case the analytic evaluation is restricted to the thermodynamic limit of the infinite-range model.

For sequential update the situation is much more complicated. In general the saddlepoint integration cannot be performed due to the lack of 'translational invariance' of the spins as it is present for the other two update rules: the first spin of the update sequence is not equivalent to any other spin of the system. For any pair of spins there exists an order in which they have to be updated depending on the starting point of the sequence. This is the main reason why there exist no analytic calculations for sequential update in the infinite-range model. Nevertheless we could perform a new kind of improved annealed approximation which yields a very good result when compared with numerical simulations (a trivial application of the annealed approximation would give a vanishing remanent energy). This new method of calculation is therefore, at least for the fully asymmetric model, a good way to get additional information on the properties of the system in the long-time limit.

## 2. Definitions

The quantity of interest, the 'energy'  $E$ , is simply defined as the long-time limit of the average alignment of the spins with their local field:

$$E(t) = -\frac{1}{2N} \sum_i \sigma_i(t) h_i(t) \quad h_i(t) = \sum_{j(\neq i)} J_{ij} \sigma_j(t) \quad (1)$$

where  $h_i(t)$  is the local field of spin  $i$ . This quantity depends on the kind of dynamics which is used: parallel (p), sequential (s) or random sequential (rs). The update rules

of the three dynamics are given as usual by

$$(p) \quad \sigma_i^t = \text{sgn} \left( \sum_{j(\neq i)} J_{ij} \sigma_j^{t-1} \right) \quad (2)$$

$$(s) \quad \sigma_i^t = \text{sgn} \left( \sum_{j=1}^{i-1} J_{ij} \sigma_j^t + \sum_{j=i+1}^N J_{ij} \sigma_j^{t-1} \right) \quad (3)$$

$$(rs) \quad \frac{d}{dt} \langle \sigma_i(t) \rangle = -\langle \sigma_i(t) \rangle + \left\langle \text{sgn} \left( \sum_{j(\neq i)} J_{ij} \sigma_j(t) \right) \right\rangle \quad (4)$$

where the brackets for random sequential update mean the average over all possible update sequences. In the discrete version of (4), which is used for numerical simulations, one chooses the (single) spin to be updated at a time step at random. Taking the time constant proportional to  $1/N$ , the differential equation (4) follows in the limit of large  $N$ . The couplings are distributed according to a bivariate distribution with moments

$$\langle J_{ij} \rangle = \langle J_{ji} \rangle = 0 \quad \langle J_{ij}^2 \rangle = \langle J_{ji}^2 \rangle = J^2 / N' \quad \langle J_{ij} J_{ji} \rangle = \lambda J^2 / N' \quad (5)$$

where  $N' = N - 1$ . Note that it is not necessary to assume a Gaussian distribution or any details of the higher moments, since only the first two moments contribute in the thermodynamic limit. We will consider exclusively the fully asymmetric case with  $\lambda = 0$  for which several exact results are known.

### 3. Parallel update

The simplest case is parallel update. The update rule (2) shows that a global transformation  $J_{ij} \rightarrow -J_{ij}$  only changes the sign of  $\sigma_i^t$  at odd time steps and leaves it unchanged at even time steps. Therefore the magnetization, averaged over all configuration of the couplings, is zero for all odd time steps, independent of the initial conditions  $\sigma_i^0$ . At even time steps the magnetization remains unchanged. It follows that the quantity  $\sigma_i^t J_{ij} \sigma_j^t$  changes its sign under the transformation of the  $J_{ij}$  independent of the parity of  $t$  and the average energy is zero for all time steps. This simple argument is valid for all kinds of lattices and system sizes. Note that it applies also for symmetric couplings.

Another consequence is that even the quantity

$$\sigma_i \text{sgn} \left( \sum_{j(\neq i)} J_{ij} \sigma_j(t) \right) \quad (6)$$

will vanish. This means that, on average, one half of the spins flip at every time step. This is the reason for the huge number of cycles of length two in the asymmetric SK model mentioned above.

### 4. Sequential update

In the case of sequential update we have used a kind of a self-consistent annealed approximation, because the functional-integral approach in the present formulation cannot be applied. Nevertheless the result is in excellent agreement with numerical simulations of the quenched model, which depend only very weakly on the system size. In the following we present the calculation of the remanent energy.

Since we want to use the annealed approximation we introduce new couplings after every run through the system, i.e. asking neuron 1 to neuron  $N$  for flipping. This is said to be one time step. Since the distribution of the couplings is even, it is possible, at every time step, to choose separately a gauge transformation

$$J_{ij} \sigma_j^{l-1} \rightarrow J_{ij}^l \quad (i < j) \quad J_{ij}^l \sigma_j^l \rightarrow J_{ij}^l \quad (i > j)$$

which does not affect the distribution (5) of the couplings.

In the following  $k = \sigma_1 h_1$  denotes the local field in the direction of neuron 1. After the alignment of neuron 1, the distribution of its field is given by

$$\mathcal{P}(k) = \Theta(k) \sqrt{\frac{2}{\pi J^2}} e^{-k^2/2J^2}. \tag{7}$$

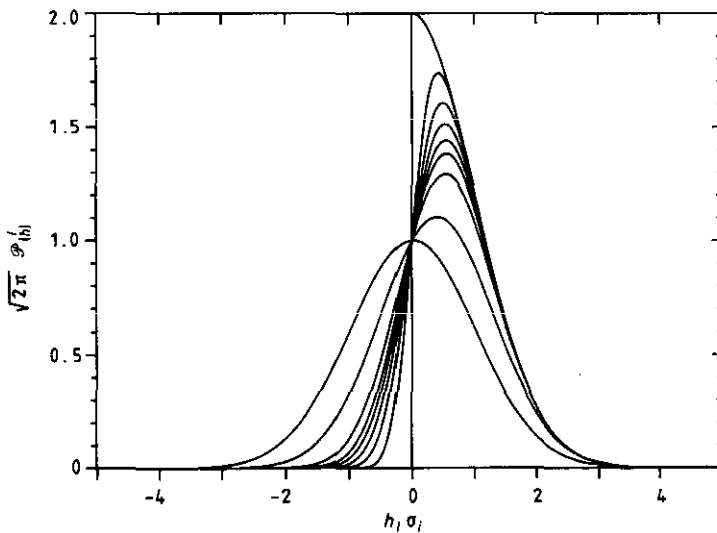
Then neurons 2 to  $N$  are updated. If during this updating exactly  $l$  ( $0 \leq l \leq N-1$ ) neurons are flipped, the new distribution  $\mathcal{P}^l(k)$  of the local field in the direction of neuron 1 is given by

$$\begin{aligned} \mathcal{P}^l(k) = & \prod_{j=2}^N \left[ \sqrt{\frac{N'}{2\pi J^2}} \int_{-\infty}^{+\infty} dJ_{1j} \exp\left(-\frac{N' J_{1j}}{2J^2}\right) \right] \\ & \times \delta\left(k - \sum_{j \notin \mathcal{F}} J_{1j} + \sum_{j \in \mathcal{F}} J_{1j}\right) 2\Theta\left(\sum_{j=2}^N J_{1j}\right) \end{aligned} \tag{8}$$

where  $\mathcal{F}$  represents the set of the  $l$  flipped neurons. The  $\Theta$ -function reflects the fact that neuron 1 points in direction of its local field. The evaluation of the integrals yields

$$\mathcal{P}^l(k) = \frac{1}{\sqrt{2\pi J^2}} \left[ 1 + \operatorname{erf}\left\{ \frac{N' - 2l}{\sqrt{8l(N' - l)J^2}} k \right\} \right] e^{-k^2/2J^2}. \tag{9}$$

This function is shown in figure 1 for different numbers  $l$  of flipped neurons. Directly after the alignment of neuron 1 the distribution of  $k$  will be always twice the positive part of a Gaussian distribution since  $\operatorname{erf}(x) = -\operatorname{erf}(-x)$ .



**Figure 1.** Distribution  $\mathcal{P}^l$  of the fields  $h_i \sigma_i$  ( $J = 1$ ) of the fully asymmetric model ( $\lambda = 0$ ) for 64 neurons and sequential update. The curves are shown for  $l = 0, 1, 2, 3, 4, 5, 7, 15, 31$ , where  $l$  denotes the number of flipped neurons. The maximum of the curves is monotonically decreasing with  $l$  for  $0 \leq l \leq N'/2$ .

The main assumption in the following is that the probability  $a$  to be flipped ('acceptance') is the same for every neuron and every time step ('equilibrium dynamics'). The probability that exactly  $l$  neurons are flipped before neuron 1 is asked again is then given by  $\binom{N'}{l} a^l (1-a)^{N'-l}$ . The probability  $f(a)$  for neuron 1 to be flipped as a function of the acceptance  $a$  can therefore be written as

$$f(a) = \sum_{l=0}^{N'} \binom{N'}{l} a^l (1-a)^{N'-l} \int_{-\infty}^0 dk P^l(k) \tag{10}$$

with

$$\int_{-\infty}^0 dk P^l(k) = \frac{1}{\pi} \cos^{-1} \left( 1 - 2 \frac{l}{N'} \right). \tag{11}$$

Since we are looking for self-consistent solutions of (9), we are interested in analysing the solutions of  $f(a) = a$ . One can simply verify that there exist three solutions:  $a = 0$  (stable state),  $a = 1$  (cycle of length 2) and  $a = \frac{1}{2}$  (half of the spins are flipping). The last solution is most easily found by means of the relation

$$\int_{-\infty}^0 dk P^l(k) = 1 - \int_{-\infty}^0 dk P^{N'-l}(k) \tag{12}$$

which is a simple consequence of (11). Starting with a neuron configuration independent of the couplings the acceptance will be  $\frac{1}{2}$  for every neuron, i.e. every neuron has a local field which is governed by a complete Gaussian distribution. This will be the case for every time step.

The stability of the above solutions can be analysed easily using the inequality  $\cos^{-1}(1-2x) > x$  for  $0 < x < \frac{1}{2}$ . It turns out that the only stable solution is  $a = \frac{1}{2}$  while the other two are unstable.

We now turn to the calculation of the mean alignment per neuron. After the flip of  $l$  neurons we obtain:

$$\frac{E^l}{N} = -\frac{1}{2} \int_{-\infty}^{+\infty} dk k P^l(k) = -\frac{J}{\sqrt{2\pi}} \left( 1 - 2 \frac{l}{N'} \right). \tag{13}$$

$E$  is then given by the average of  $E^l$  over one period

$$E = \frac{1}{N} \sum_{i=0}^{N'} \sum_{l=0}^i \binom{i}{l} a^l (1-a)^{N'-l} E^l. \tag{14}$$

For the three solutions of (9) one calculates

$$\frac{E}{N} = -\frac{J}{\sqrt{2\pi}} = -0.3989J \quad a = 0 \tag{15}$$

$$\frac{E}{N} = 0 \quad a = 1 \tag{16}$$

$$\begin{aligned} \frac{E}{N} &= -\frac{J}{\sqrt{2\pi}} \frac{1}{N} \sum_{i=0}^{N'} \frac{1}{2^i} \sum_{l=0}^i \binom{i}{l} \left( 1 - 2 \frac{l}{N'} \right) \\ &= -\frac{J}{\sqrt{8\pi}} = -0.1995J \quad a = \frac{1}{2}. \end{aligned} \tag{17}$$

Our numerical simulations of systems up to 1024 neurons give an extrapolated value of  $E_s = -(0.1990 \pm 0.0005)J$  and we found it to depend only very weakly on the system size. As mentioned above, the only stable solution corresponds to  $a = \frac{1}{2}$ , which is verified very well by the simulations. Thus we get the same picture as for parallel dynamics: half of the spin, on average, are flipping during a run through the system. But here the local field of a flipping spin varies continuously as the others are updated whereas for parallel update only two different values are possible for every spin. This gives the difference in the resulting energies.

It is an interesting fact that the calculated value of the remanent energy seems to be the exact one, although the annealed approximation used above does not take into account the full correlation in time. On the other hand it is known that the additional randomness introduced through the fully asymmetric couplings simplifies the properties of the model in such a way that annealed (tree) approximations might give right answers for the fully connected model [13, 14].

### 5. Random-sequential update

In the case of random-sequential update we have used the path-integral formulation introduced by Sommers [11, 12]. The starting point is a generating functional for spin and field configurations given by [7]

$$\mathcal{Z}[i\hat{\sigma}, i\hat{h}] = \left\langle \exp \left( \int i\hat{\sigma} \mathcal{M} + \int i\hat{h} \varphi \right) \right\rangle_{\varphi} \quad (18)$$

where

$$\mathcal{M} = M_0 e^{-(t-t_0)} + \int_{t_0}^t d\tau e^{-(t-\tau)} [\tanh(\beta\varphi(\tau)) + i\hat{\sigma}(1 - M^2(\tau))] \quad (19)$$

and  $\varphi(\tau)$  denotes a Gaussian stochastic process with

$$\langle \varphi(t) \rangle_{\varphi} = 0 \quad \text{and} \quad \langle \varphi(t)\varphi(t') \rangle_{\varphi} = J^2 C(t, t'). \quad (20)$$

The averaged autocorrelation function

$$C(t, t') = \frac{1}{N} \sum_{i=1}^N \overline{\langle \sigma_i(t)\sigma_i(t') \rangle} \quad (21)$$

has to be determined self-consistently via

$$C(t, t') = \frac{\delta}{\delta i\hat{\sigma}(t)} \frac{\delta}{\delta i\hat{\sigma}(t')} \mathcal{Z}[i\hat{\sigma}, i\hat{h}] \Big|_{\hat{\sigma}=0, \hat{h}=0} \quad (22)$$

as was done in [7]. The energy is now given by the following spin-field correlation

$$\langle \sigma(t)h(t) \rangle = \frac{\delta}{\delta i\hat{\sigma}(t)} \frac{\delta}{\delta i\hat{h}(t)} \mathcal{Z}[i\hat{\sigma}, i\hat{h}] \Big|_{\hat{\sigma}=0, \hat{h}=0} \quad (23)$$

and therefore

$$E(t) = -\frac{1}{2} \int_{t_0}^t d\tau e^{-(t-\tau)} \langle \varphi(t) \tanh \beta\varphi(\tau) \rangle_{\varphi}. \quad (24)$$

A straightforward calculation yields

$$E(t) = -J\gamma \int_{t_0}^t d\tau e^{-(t-\tau)} C(t, \tau) \quad (25)$$

with

$$\gamma = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \beta J [\cosh(\beta J z)]^{-2}. \quad (26)$$

In the limit  $t \rightarrow \infty$  the system is in a stationary state (i.e.  $C(t + \Delta t, t) = C(\Delta t)$ ), and one gets for the 'equilibrium' value of the energy  $E = \lim_{t \rightarrow \infty} E(t)$ :

$$E = -J\gamma \int_0^\infty d\tau e^{-\tau} C(\tau). \quad (27)$$

For  $\beta \rightarrow \infty$  ( $T=0$ ) it is  $\gamma = 1/\sqrt{2\pi}$  and with the help of (21) and (22) in [7] we can transform the integral and get

$$E = -\frac{J}{\sqrt{2\pi}} \int_0^1 dy y (\sqrt{g(y)})^{-1} \exp\left(-\int_y^1 dx (\sqrt{g(x)})^{-1}\right) \quad (28)$$

with

$$g(x) = x^2 + \frac{4}{\pi} [1 - \sin^{-1}(x) - \sqrt{1-x^2}]. \quad (29)$$

The numerical evaluation of this integral yields

$$E_{rs} = -(0.254\,764\,72 \pm 10^{-8})J \quad (30)$$

which is in accordance with the result from the computer simulations  $E_{rs} = -(0.2548 \pm 0.0005)J$ . From the derivation of this result one can conclude that it is non-trivial and cannot be derived by an annealed approximation, since in (25) the complete autocorrelation function in time has to be inserted.

## 6. Discussion

In this paper we have shown that in the fully asymmetric SK model it is possible to calculate analytically the energy for the three different kinds of update procedures (parallel, sequential and random sequential) with three different methods. The most powerful is the functional-integral method which is applicable for random sequential and up to some limits also for parallel update, but it fails for random sequential update. This is a troublesome fact since from a numerical point of view sequential dynamics is mostly and probably easiest studied, at least on a sequential computer. So in this case only a self-consistent annealed approximation could be performed (the simple annealed approximation would yield trivial results), but the value of the remanent energy is in very good agreement with our numerical simulations. It was mentioned above that this observation is not new for fully asymmetric couplings. As expected the value for the random sequential update is the lowest one. For deterministic dynamics like sequential and parallel the system remains in a state with higher remanent energy.

Monte Carlo simulations in the SK model with symmetric couplings and with sequential update yield a value of  $E = -(0.704 \pm 0.007)J$  [15] for the remanent energy which is much lower than all the above ones (of course different from the ground state energy which corresponds to the absolute minimum of the energy surface).



This is due to the fact that there is no real 'freezing' process in the systems with fully asymmetric couplings. As mentioned in the introduction this kind of analytical calculation is not possible for finite  $\lambda$  (see (5)), but perhaps can be studied by approximative methods in the future.

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